Logical Equivalence,   3
Logical Truths,
and Contradictions

3-1. LOGICAL EQUIVALENCE

I introduced logic as the science of arguments. But before turning to arguments, we need to extend and practice our understanding of logic's basic tools as I introduced them in chapter 1. For starters, let's look at the truth table for 'A', '~A', and the negation of the negation of 'A', namely, '~~A'

A	~A	~~A
t	f	t
f	t	f

This truth table exhibits the special situation which I mentioned at the end of the last chapter: The truth value of '~~A' is always the same as that of 'A'. Logicians say that 'A' and '~~A' are *Logically Equivalent.*

As we will see in a moment, much more complicated sentences can be logically equivalent to each other. To get clear on what this means, let us review some of the things that truth tables do for us. Suppose we are looking at a compound sentence, perhaps a very complicated one which uses many sentence letters. When we write out the truth table for such a

sentence, we write out all the possible cases, that is, all the possible assignments of truth values to sentence letters in all possible combinations. In each one of these possible cases our original sentence has the truth value t or the truth value f.

Now suppose that we look at a second sentence which uses the same sentence letters, or perhaps only some of the sentence letters that the first sentence uses, and no new sentence letters. We say that the two sentences are logically equivalent if in each possible case, that is, for each line of the truth table, they have the same truth value.

> Two sentences of sentence logic are *Logically Equivalent* if and only if in each possible case (for each assignment of truth values to sentence letters) the two sentences have the same truth value.

What we said about the double negation of 'A' naturally holds quite generally:

> The *Law of Double Negation* (DN): For any sentence **X**, **X** and ~~**X** are logically equivalent.

Here are two more laws of logical equivalence:

> *De Morgan's Laws* (DM): For any sentences **X** and **Y**, ~(**X**&**Y**) is logically equivalent to ~**X**v~**Y**. And ~(**X**v**Y**) is logically equivalent to ~**X**&~**Y**.

Thus 'Adam is not both ugly and dumb.' is logically equivalent to 'Either Adam is not ugly or Adam is not dumb.' And 'Adam is not either ugly or dumb.' is logically equivalent to 'Adam is not ugly and Adam is not dumb.' You should check these laws with truth tables. But I also want to show you a second, informal way of checking them which allows you to "see" the laws. This method uses something called *Venn Diagrams*.

A Venn diagram begins with a box. You are to think of each point inside the box as a possible case in which a sentence might be true or false. That is, think of each point as an assignment of truth values to sentence letters, or as a line of a truth table. Next we draw a circle in the box and label it with the letter '**X**', which is supposed to stand for some arbitrary sentence, atomic or compound. The idea is that each point inside the circle represents a possible case in which **X** is true, and each point outside the circle represents a possible case in which **X** is false.

Look at Figure 3–1. What area represents the sentence ~**X**? The area **outside** the circle, because these are the possible cases in which **X** is false.

Now let's consider how Venn diagrams work for compound sentences built up from two components, **X** and **Y**. Depending on what the sentences **X** and **Y** happen to be, both of them might be true, neither might be true, or either one but not the other might be true. Not to omit any of
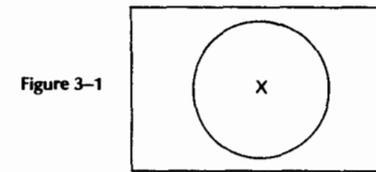
Figure 3–1

these eventualities, we must draw the circles representing **X** and **Y** as overlapping, as in Figure 3–2 and 3–3.

X & Y

X v Y
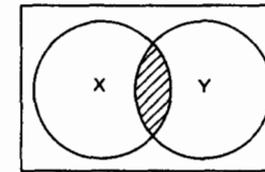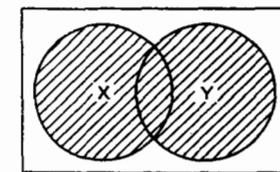
Figure 3–2   Figure 3–3

The conjunction **X**&**Y** is true in just those cases represented by points that lie inside both the **X** and **Y** circles, that is, the shaded area in Figure 3–2. The disjunction **X**v**Y** is true in just those cases represented by points that lie inside either the **X** or the **Y** circle (or both), that is, the shaded area in Figure 3–3.

Now we can use Venn diagrams to check De Morgan's laws. Consider first a negated conjunction. Look for the area, shown in Figure 3–4, which represents ~(**X**&**Y**). This is just the area **outside** the shaded lens in Figure 3–2.
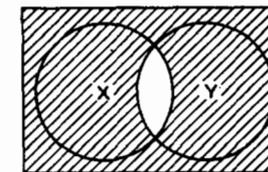
Figure 3–4

~(X & Y)

Let us compare this with the area which represents ~**X**v~**Y**. We draw overlapping **X** and **Y** circles. Then we take the area outside the first circle (which represents ~**X**; see Figure 3–5), and we take the area outside the second (which represents ~**Y**; see Figure 3–6).

Finally, we put these two areas together to get the area representing the disjunction ~**X**v~**Y**, as represented in Figure 3–7.

Notice that the shaded area of Figure 3–7, representing ~**X**v~**Y**, is the same as that of Figure 3–4, representing ~(**X**&**Y**). The fact that the same
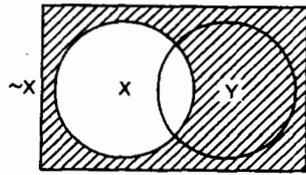
Figure 3–5
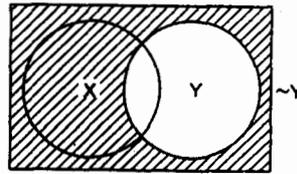
Figure 3–6
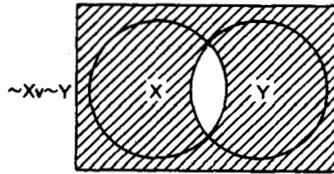
Figure 3–7

shaded area represents both ~Xv~Y and ~(X&Y) means that the two sentences are true in exactly the same cases and false in the same cases. In other words, they always have the same truth value. And that is just what we mean by two sentences being logically equivalent.

Now try to prove the other of De Morgan's laws for yourself using Venn diagrams.

Here are two more laws of logical equivalence:

> The *Distributive Laws:* For any three sentences, **X, Y,** and **Z, X&(YvZ)** is logically equivalent to **(X&Y)v(X&Z)**. And **Xv(Y&Z)** is logically equivalent to **(XvY)&(XvZ)**.

For example, 'Adam is both bold and either clever or lucky.' comes to the same thing as 'Adam is either both bold and clever or both bold and lucky.' You should prove these two laws for yourself using Venn diagrams. To do so, you will need a diagram with three circles, one each representing **X, Y,** and **Z.** Again, to make sure that you omit no possible combination of truth values, you must draw these so that they all overlap, as in Figure 3–8.
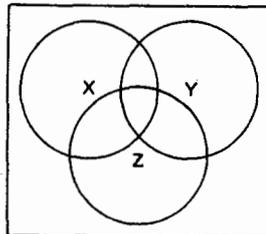
Figure 3–8

Fill in the areas to represent YvZ, and then indicate the area which represents the conjunction of this with **X.** In a separate diagram, first fill in the areas representing **X&Y** and **X&Z,** and then find the area corresponding to the disjunction of these. If the areas agree, you will have demonstrated logical equivalence. Do the second of the distributive laws similarly. Also, if you feel you need more practice with truth tables, prove these laws using truth tables.

EXERCISES

3–1. Prove the second of De Morgan's laws and the two distributive laws using Venn diagrams. Do this in the same way that I proved the first of De Morgan's laws in the text, by drawing a Venn diagram for each proof, labeling the circles in the diagram, and explaining in a few sentences how the alternate ways of getting the final area give the same result. Use more than one diagram if you find that helpful in explaining your proof.

3–2. SUBSTITUTION OF LOGICAL EQUIVALENTS AND SOME MORE LAWS

We can't do much with our laws of logical equivalence without using a very simple fact, which our next example illustrates. Consider

(1)   ~~AvB.

'~~A' is logically equivalent to 'A'. This makes us think that (1) is logically equivalent to

(2)   AvB.

This is right. But it is important to understand why this is right. A compound sentence is made up of component sentences, which in turn may be made up of further component sentences. How do subsentences (components, or components of components, or the like) affect the truth value of the original sentence? **Only** through their truth values. The only way that a subsentence has any effect on the truth values of a larger sentence is through the subsentence's truth value. (This, again, is just what we mean by saying that compound sentences are truth functions.) But if only the truth values matter, then substituting another sentence which always has the same truth value as the first can't make any difference.

I'll say it again in different words: Suppose that **X** is a subsentence of some larger sentence. Suppose that **Y** is logically equivalent to **X**, which means that **Y** and **X** always have the same truth value. **X** affects the truth value of the larger sentence only through its (i.e., **X**'s) truth value. So, if we substitute **Y** for **X**, there will be no change in the larger sentence's truth value.

But this last fact is just what we need to show our general point about logical equivalence. The larger sentence will have the same truth value before and after the substitution; that is, the two versions of the larger sentence will be logically equivalent:

> The *Law of Substitution of Logical Equivalents* (SLE): Suppose that **X** and **Y** are logically equivalent, and suppose that **X** occurs as a subsentence of some larger sentence **Z**. Let **Z*** be the new sentence obtained by substituting **Y** for **X** in **Z**. Then **Z** is logically equivalent to **Z***.

Let's apply these laws to an example. Starting with the sentence

$$\sim[(\sim A v \sim B) \& (\sim A v B)]$$

we can apply one of De Morgan's laws. This sentence is the negation of a conjunction, with the conjuncts '$\sim A v \sim B$' and '$\sim A v B$'. De Morgan's law tells us that this first line is logically equivalent to the disjunction of the negation of the two original conjuncts:

$$\sim(\sim A v \sim B) v \sim(\sim A v B) \qquad \text{DM}$$

(The 'DM' on the right means that this line was obtained from the previous line by applying one of De Morgan's laws.)

Did you have trouble understanding that one of De Morgan's laws applies to the sentence? If so, try using the idea of the main connective introduced in chapter 1. Ask yourself: "In building this sentence up from its parts, what is the last thing I do?" You apply the negation sign to '$(\sim A v \sim B) \& (\sim A v B)$'. So you know the original sentence is a negation. Next, ask yourself, what is the last thing I do in building '$(\sim A v \sim B) \& (\sim A v B)$' up from **its** parts? Conjoin '$\sim A v \sim B$' with '$\sim A v B$'. So '$(\sim A v \sim B) \& (\sim A v B)$' is a conjunction. The original sentence, then, is the negation of a conjunction, that is, a sentence of the form $\sim(X \& Y)$, where, in our example, **X** is the sentence '$\sim A v \sim B$' and **Y** is the sentence '$\sim A v B$'. Applying De Morgan's law to $\sim(X \& Y)$ gives $\sim X v \sim Y$; in other words, in our example, '$\sim(\sim A v \sim B) v \sim(\sim A v B)$'.

·Next, we can apply De Morgan's law to each of the components, '$\sim(\sim A v \sim B)$' and '$\sim(\sim A v B)$', and then use the law of substitution of logical equivalents to substitute the results back into the full sentence. Doing this, we get

$$(\sim\sim A \& \sim\sim B) v (\sim\sim A \& \sim B) \qquad \text{DM, SLE}$$

(As before, 'DM' on the right means that we have used one of De Morgan's laws. 'SLE' means that we have also used the law of substitution of logical equivalents in getting the last line from the previous one.)

Now we can apply the law of double negation (abbreviated 'DN') to '$\sim\sim A$' and to '$\sim\sim B$' and once more substitute the results into the larger sentence. This gives

$$(A \& B) v (A \& \sim B) \qquad \text{DN, SLE}$$

We have only one more step to do. If you look carefully, you will see that the distributive law (abbreviated 'D') applies to the last line. So the last line is logically equivalent to

$$A \& (B v \sim B) \qquad \text{D}$$

This might not be clear at first. As I stated the distributive law, you might think it applies only to show that the very last line is logically equivalent to the next to last line. But if **X** is logically equivalent to **Y**, then **Y** is logically equivalent to **X**! Logical equivalence is a matter of always having the same truth value, so if two sentences are logically equivalent, it does not matter which one gets stated first. Often students only think to apply a law of logical equivalents in the order in which it happens to be stated. But the order makes no difference—the relation of logical equivalence is symmetric, as logicians say.

Let's put all the pieces of this problem together. In the following summary, each sentence is logically equivalent to the previous sentence and the annotations on the right tell you what law or laws give you a line from the previous one.

$$\sim[(\sim A v \sim B) \& (\sim A v B)]$$
$$\sim(\sim A v \sim B) v \sim(\sim A v B) \qquad \text{DM}$$
$$(\sim\sim A \& \sim\sim B) v (\sim\sim A \& \sim B) \qquad \text{DM, SLE}$$
$$(A \& B) v (A \& \sim B) \qquad \text{DN, SLE}$$
$$A \& (B v \sim B) \qquad \text{D}$$

Actually, all I have really proved is that each of the above sentences is logically equivalent to the next. I really want to show that the first is logically equivalent to the last. Do you see why that must be so? Because being logically equivalent just means having the same truth value in all possible cases, we trivially have

> The *Law of Transitivity of Logical Equivalence* (TLE): For any sentences **X**, **Y**, and **Z**, if **X** is logically equivalent to **Y** and **Y** is logically equivalent to **Z**, then **X** is logically equivalent to **Z**.

Repeated use of this law allows us to conclude that the first sentence in our list is logically equivalent to the last. Many of you may find this point obvious. From now on, transitivity of logical equivalence will go without saying, and you do not need explicitly to mention it in proving logical equivalences.

Here are some more easy, but very important, laws:

> The *Commutative Law* (CM):   For any sentences **X** and **Y**, **X&Y** is logically equivalent to **Y&X**. And **XvY** is logically equivalent to **YvX**.

In other words, order in conjunctions and disjunctions does not make a difference. Note that the commutative law allows us to apply the distributive law from right to left as well as from left to right. For example, '(A&B)vC' is logically equivalent to '(AvC)&(BvC)'. You should write out a proof of this fact using the commutative law and the distributive law as I stated it originally.

Next, the *Associative Law* tells us that 'A&(B&C)' is logically equivalent to '(A&B)&C'. To check this, try using a Venn diagram, which in this case gives a particularly quick and clear verification. Or simply note that both of these sentences are true only when 'A', 'B', and 'C' are all true, and are false when one or more of the sentence letters are false. This fact shows that in this special case we can safely get away with dropping the parentheses and simply writing 'A&B&C', by which we will mean either of the logically equivalent 'A&(B&C)' or '(A&B)&C'. Better yet, we will extend the way we understand the connective '&'. We will say that '&' can appear between any number of conjuncts. The resulting conjunction is true just in case all of the conjuncts are true, and the conjunction is false in all other cases.

The same sort of generalization goes for disjunction. 'Av(BvC)' is logically equivalent to '(AvB)vC'. Both of these are true just in case one or more of 'A', 'B', and 'C' are true and false only if all three of 'A', 'B', and 'C' are false. (Again, a Venn diagram provides a particularly swift check.) We extend our definition of 'v' so that it can appear between as many disjuncts as we like. The resulting disjunction is true just in case at least one of the disjuncts is true and the disjunction is false only if all the disjuncts are false.

> The *Associative Law* (A):   For any sentences **X**, **Y**, and **Z**, **X&(Y&Z)**, **(X&Y)&Z**, and **X&Y&Z** are logically equivalent to each other. And **Xv(YvZ)**, **(XvY)vZ**, and **XvYvZ** are logically equivalent to each other. Similarly, conjunctions with four or more components may be arbitrarily grouped and similarly for disjunctions with four or more disjuncts.

Here is yet another easy law. Clearly, **X&X** is logically equivalent to **X**. Likewise, **XvX** is logically equivalent to **X**.

The *Law of Redundancy* (RD):   For any sentence **X**, **X&X** is logically equivalent to **X**. Similarly, **XvX** is logically equivalent to **X**.

Let us apply this law in a little example. Again, each line is logically equivalent to the next (RD stands for the law of redundancy):

```
~(A&B)&(~Av~B)
(~Av~B)&(~Av~B)      DM, SLE
~Av~B                RD
```

Before asking you to practice these laws, let me give you a more extended example which illustrates all the laws I have introduced so far:

```
~(Av~B)v[(CvB)&(Cv~A)]
~(Av~B)v[Cv(B&~A)]        D, SLE
~(Av~B)v[(B&~A)vC]        CM, SLE
[~(Av~B)v(B&~A)]vC        A
[~(Av~B)v(~A&B)]vC        CM, SLE
[~(Av~B)v(~A&~~B)]vC      DN, SLE
[~(Av~B)v~(Av~B)]vC       DM, SLE
~(Av~B)vC                 RD, SLE.
```

EXERCISES

3-2. Prove the following logical equivalences. Write out your proofs as I did in the text specifying which laws you use in getting a line from the previous line. You can use the abbreviations for the laws found in the text. Until you feel comfortable with the easy laws, please include all steps. But when they begin to seem painfully obvious, you may combine the following laws with other steps and omit mentioning that you have used them: double negation, the associative law, the commutative law, and the law of substitution of logical equivalents. You must explicitly specify any other law you use.

a)   'Bv~A' is logically equivalent to '~(A&~B)'.

b)   '(A&B)vC' is logically equivalent to '(AvC)&(BvC)'. (Show all steps in this problem.)

c)   'A&(~~CvB)' is logically equivalent to '(A&C)v(A&B)'.

d)   '~[(A&~B)v(C&~B)]' is logically equivalent to '(~A&~C)vB'.

e)   '(AvB)&(CvD)' is logically equivalent to '(A&C)v(B&C)v(A&D)v(B&D)'.

f)   '(A&B)v(C&D)' is logically equivalent to '(AvC)&(BvC)&(AvD)&(BvD)'.

g)   '(C&A)v(B&C)v[C&~(~B&~A)]' is logically equivalent to 'C&(AvB)'.

h)   'C&~A' is logically equivalent to 'C&[~Av~(~CvA)]'.

i) '~A&B&C' is logically equivalent to
'C&{~(Av~B)v[B&~(~CvA)]}'.

3-3. Give a formal statement of De Morgan's laws in application to negations of conjunctions and disjunctions with three components. Model your formal statement on the formal statement in the text. It should begin as follows:

De Morgan's Laws: For any sentences **X, Y,** and **Z** . . .

3-3. LOGICAL TRUTHS AND CONTRADICTIONS

Let us look at another interesting example:

A	~A	Av~A
t	f	t
f	t	t

'Av~A' is true no matter what. Such a sentence is called a *Logical Truth.*

> A sentence of sentence logic is a *Logical Truth* just in case it is true in all possible cases, that is, just in case it is true for all assignments of truth values to sentence letters.

Many authors use the word *Tautology* for a logical truth of sentence logic. I prefer to use the same expression, 'logical truth', for this idea as it applies in sentence logic and as it applies to predicate logic, which we will study in volume II.

Clearly there will also be sentences which are false no matter what, such as

A	~A	A&~A
t	f	f
f	t	f

Such a sentence is called a *Contradiction.*

> A sentence of sentence logic is a *Contradiction* just in case it is false in all possible cases, that is, just in case it is false for all assignments of truth values to sentence letters.

Later on in the course, logical truths and contradictions will concern us quite a bit. They are interesting here because they provide several further laws of logical equivalence:

The *Law of Logically True Conjunct* (LTC): If X is any sentence and Y is any logical truth, then X&Y is logically equivalent to X.

The *Law of Contradictory Disjunct* (CD): If X is any sentence and Y is any contradiction, then XvY is logically equivalent to X.

You should be able to show that these laws are true. Furthermore, you should satisfy yourself that a conjunction is always a contradiction if one of its conjuncts is a contradiction and that a disjunction is always a logical truth if one of its disjuncts is a logical truth.

EXERCISES

3-4. Explain why a disjunction is always a logical truth if one of its disjuncts is a logical truth. Explain why a conjunction is always a contradiction if one of its conjuncts is a contradiction.

3-5. Further simplify the sentence 'A&(Bv~B)', which was the last line of the first example in section 3-2.

3-6. Prove the following logical equivalences, following the same instructions as in exercise 3-2:

a) 'A&(~AvB)' is logically equivalent to 'A&B'.
b) 'AvB' is logically equivalent to 'Av(~A&B)'.
c) 'A' is logically equivalent to '(A&B)v(A&~B)'. (This equivalence is called the *Law of Expansion.* You may find it useful in some of the other problems.)
d) 'A' is logically equivalent to '(AvB)&(Av~B)'.
e) 'A&[Bv(~A&C)]' is logically equivalent to 'A&B'.
f) 'CvB' is logically equivalent to '(C&A)v(B&A)v(C&~A)v(B&~A)'.
g) 'C&B' is logically equivalent to '(CvA)&(Bv~D)&(~AvC)&(DvB)'.
h) '(A&B)v(~A&~B)' is logically equivalent to '(~AvB)&(~BvA)'.
i) '~Av~BvC' is logically equivalent to '~(~AvB)v~AvC'.

3-7. For each of the following sentences, determine whether it is a logical truth, a contradiction, or neither. (Logicians say that a sentence which is neither a logical truth nor a contradiction is *Contingent,* that is, a sentence which is true in some cases and false in others.) Simplify the sentence you are examining, using the laws of logical equivalence, to show that the sentence is logically equivalent to a sentence you already know to be a logical truth, a contradiction, or neither.

a) (B&A)v(B&~A)
b) B&[(~AvA)&~B)]

c) (A&B)v[(A&~B)&(~A&B)]
d) (AvB)v~(A&C)
e) (AvB)&(Av~B)&(~AvB)&(~Av~B)
f) ~A&~B&(AvB)
g) (AvC)&C&(Av~C)
h) (A&B)v~Bv~A
i) (~B&A)v(~A&B)v(A&B)v(~B&~A)
j) (Av~AvB)&(Av~Av~C)
k) (A&C)v(A&~C)v(B&C)v(B&~C)
l) ~(~AvC)v~(~BvC)v~(AvB)vC

3-4. DISJUNCTIVE NORMAL FORM AND THE SHEFFER STROKE

Now that we understand logical equivalence, we can use it to put any sentence into a form which shows very clearly what the sentence says. As usual, we will start by looking at an example. Start with the truth table for Av~B:

	A	B	~B	Av~B
Case 1	t	t	f	t
Case 2	t	f	t	t
Case 3	f	t	f	f
Case 4	f	f	t	t

The truth table tells us that 'Av~B' is true in cases 1, 2, and 4. We can easily say what case 1 says using a sentence of sentence logic. Case 1 just says that 'A' and 'B' are both true, which we can say with 'A&B'. In the same way, case 2 says that 'A' is true and 'B' is false, which we say in sentence logic with 'A&~B'. Finally, '~A&~B' says that case 4 holds, that is, that 'A' is false and 'B' is false. Of course, none of these things says what 'Av~B' says, which was that either case 1 is true or case 2 is true or case 4 is true. But, clearly, we can say this in sentence logic by using the disjunction of the three sentences, each one of which describes one of the cases covered by 'Av~B'. That is

'Av~B' is logically equivalent to '(A&B)v(A&~B)v(~A&~B)'.

'(A&B)v(A&~B)v(~A&~B)' is said to be in *Disjunctive Normal Form*, and it says that **either** 'A' and 'B' are both true **or** 'A' is true and 'B' is false **or** 'A' is false and 'B' is false. This disjunction is logically equivalent to 'Av~B' because the disjunction says just what 'Av~B' says, as shown by its truth table.

Here is a slightly different way of putting the same point. The truth table shows us the possible cases in which the sentence under study will be true. We can always write a logically equivalent sentence in disjunctive normal form by literally writing out the information contained in the truth table. For each case in which the original sentence comes out true, write the conjunction of sentence letters and negated sentence letters which describe that case. Then take the disjunction of these conjunctions. This sentence will be true in exactly those cases described by the disjuncts (in our example, in the cases described by 'A&B', by 'A&~B', and by '~A&~B'). But the original sentence is true in just these same cases—that is what its truth table tells us. So the sentence in disjunctive normal form is true in exactly the same cases as the original, which is to say that the two are logically equivalent.

I want to give you a formal summary description of disjunctive normal form. But first we must deal with three troublesome special cases. I will neutralize these troublesome cases with what may at first strike you as a very odd trick.

Consider the atomic sentence 'A'. As I have so far defined disjunctions and conjunctions, 'A' is neither a disjunction nor a conjunction. But because 'A' is logically equivalent to 'AvA', it will do no harm if we extend the meaning of 'disjunction' and say that, in a trivial way, 'A' will also count as a disjunction—that is, as a degenerate disjunction which has only one disjunct.

We can play the same trick with conjunctions. 'A' is logically equivalent to 'A&A'. So we do no harm if we extend the meaning of 'conjunction' and say that, in a trivial way, 'A' also counts as a conjunction—the degenerate conjunction which has only one conjunct.

Finally, we will want to say the same thing, not just about atomic sentence letters, but about any sentence, **X**, atomic or compound. Whatever the form of **X**, we can always view **X** as a degenerate disjunction with just one disjunct or as a degenerate conjunction with just one conjunct.

What is the point of this apparently silly maneuver? I am working toward giving a definition of disjunctive normal form which will work for **any** sentence. The idea of disjunctive normal form is that it involves a disjunction of conjunctions. But what should we say, for example, is the disjunctive normal form of the sentence 'A&B'? If we allow degenerate disjunctions with one disjunct, we can say that 'A&B' is already in disjunctive normal form—think of it as '(A&B)v(A&B)'. Again, what should we say is the disjunctive normal form of 'A'? Let's count 'A' as a degenerate conjunction with just one conjunct (think of 'A&A') and let's count this conjunction as a degenerate disjunction, as in the last example. So we can say that 'A' is already in disjunctive normal form and still think of disjunctive normal form as a disjunction of conjunctions.

We still have to discuss one more special case. What should we say is

the disjunctive normal form of a contradiction, such as 'A&~A'? We will allow repetitions of sentence letters with and without negation signs, so that, again, 'A&~A' will itself already count as being in disjunctive normal form.

Now we can say very simply:

> A sentence is in *Disjunctive Normal Form* if it is a disjunction, the disjuncts of which are themselves conjunctions of sentence letters and negated sentence letters. In this characterization we allow as a special case that a disjunction may have only one disjunct and a conjunction may have only one conjunct.
>
> For any sentence, **X**, of sentence logic, the disjunctive normal form of **X** is given by a sentence **Y** if **Y** is in disjunctive normal form and is logically equivalent to **X**. Except for contradictions, the disjunctive normal form of a sentence is the sentence's truth table expressed in sentence logic.

The fact that every sentence of sentence logic is logically equivalent to a sentence in disjunctive normal form helps to show something interesting about the connectives. All our sentences are put together using '&', 'v', and '~'. But are these connectives all we really need? Could we say new things if we added new connectives? The answer is no, if we limit ourselves to sentences which can be given in terms of a truth table. Because we can write any truth table in disjunctive normal form, using only '&', 'v' and '~', anything which we can express using a truth table we can express using just these three connectives. In other words, '&', 'v', and '~' are enough if we limit ourselves to a logic all the sentences of which are truth functions of atomic sentence letters. We say that '&', 'v', and '~' are, together, *Expressively Complete*. For given the truth table of any sentence which we might want to write, we can always write it with a sentence in disjunctive normal form.

Even more interestingly, '&', 'v' and '~' are more than we need. Using De Morgan's laws and double negation, we can always get rid of a conjunction in favor of a disjunction and some negation signs. And we can always get rid of a disjunction in favor of a conjunction and some negation signs. (Do you see how to do this?) Thus any sentence which can be represented by a truth table can be expressed using just '&' and '~'. And any such sentence can be expressed using just 'v' and '~'. So '&' and '~' are expressively complete, and 'v' and '~' are also expressively complete.

We have just seen that anything that can be represented with truth tables can be expressed with a sentence using just two connectives. Could we make do with just **one** connective? Clearly, we can't make do with just '&', with just 'v', or with just '~'. (Can you see why?) But perhaps we could introduce a new connective which can do everything all by itself. Consider the new connective '|', called the *Sheffer Stroke*, defined by

X	Y	X|Y
t	t	f
t	f	f
f	t	f
f	f	t

Work out the truth table and you will see that X|X is logically equivalent to ~X. Similarly, you can prove that (X|Y)|(X|Y) is logically equivalent to XvY. With this new fact, we can prove that '|' is expressively complete. We can express any truth function in disjunctive normal form. Using De Morgan's law and the law of double negation, we can get rid of the '&'s in the disjunctive normal form. So we can express any truth function using just 'v' and '~'. But now for each negation we can substitute a logically equivalent expression which uses just '|'. And for each disjunction we can also substitute a logically equivalent expression which uses just '|'. The final result uses '|' as its only connective. Altogether, the sentence in disjunctive normal form has been transformed into a logically equivalent sentence using just '|'. And because any truth function can be put in disjunctive normal form, we see that any truth function, that is, any sentence which could be given a truth table definition, can be expressed using just '|'.

The important idea here is that of expressive completeness:

> A connective, or set of connectives, is *Expressively Complete* for truth functions if and only if every truth function can be represented using just the connective or connectives.

Actually, the really important idea is that of a truth function. Understanding expressive completeness for truth functions will help to make sure you have the idea of a truth function clearly in mind.

EXERCISES

3–8. Put the following sentences in disjunctive normal form. You can do this most straightforwardly by writing out truth tables for the sentences and then reading off the disjunctive normal form from the truth tables. Be sure you know how to work the problems this way. But you might have more fun trying to put a sentence in disjunctive normal form by following this procedure: First, apply De Morgan's laws to drive all negations inward until negation signs apply only to sentence letters. Then use other laws to get the sentence in the final disjunctive normal form.

a) ~(A&B)

b) ~[(A&B)v(~A&C)]

3–9. Suppose you are given a sentence in which 'v' occurs. Explain in general how you can write a logically equivalent sentence in which 'v' does not occur at all. Similarly, explain how a sentence in which '&' occurs can be replaced by a logically equivalent sentence in which '&' does not occur. (*Hint:* You will need to appeal to De Morgan's laws.)

3–10. Define a new connective, '*', as representing the following truth function:

	X	Y	X*Y
case 1	t	t	f
case 2	t	f	t
case 3	f	t	t
case 4	f	f	t

Show that '*' is expressively complete.

3–11. Show that '&' is not expressively complete. That is, give a truth function and show that this truth function cannot be expressed by using '&' as the only connective. Similarly, show that 'v' is not expressively complete and show that '~' is not expressively complete. (You may find this problem hard, but please take a few minutes to try to work it.)

CHAPTER SUMMARY EXERCISE

Once again, you will find below the important terms which I have introduced in this chapter. Make sure you understand all of them by writing out a short explanation of each. You should refer to the text to make sure that you have correctly explained each term. Please keep your explanations of these terms in your notebook for reference and review.

a) Logical Equivalence
b) Venn Diagram
c) Law of Double Negation
d) De Morgan's Laws
e) Distributive Laws
f) Law of Substitution of Logical Equivalents
g) Law of Transitivity of Logical Equivalence

h) Commutative Laws
i) Associative Law
j) Law of Redundancy
k) Logical Truth
l) Contradiction
m) Law of Logically True Conjunct
n) Law of Contradictory Disjunct

If you have read section 3–4, also explain

o) Disjunctive Normal Form
p) Expressively Complete
q) Sheffer Stroke