Validity and Conditionals

4

4-1. VALIDITY

Consider the following argument:

AvB	Adam loves Eve or Adam loves Bertha.
~A	Adam does not love Eve.
B	Adam loves Bertha.

If you know, first of all, that either 'A' or 'B' is true, and in addition you know that 'A' itself is false; then clearly, 'B' has to be true. So from 'AvB' and '~A' we can conclude 'B'. We say that this argument is *Valid,* by which we mean that, without fail, if the premises are true, then the conclusion is going to turn out to be true also.

Can we make this idea of validity more precise? Yes, by using some of the ideas we have developed in the last three chapters. (Indeed one of the main reasons these ideas are important is that they will help us in making the notion of validity very precise.) Let us write out a truth table for all the sentences appearing in our argument:

	A	B	~A	AvB
case 1	t	t	f	t
case 2	t	f	f	t
case 3	f	t	t	t
case 4	f	f	t	f

We know that cases 1 through 4 constitute all the ways in which any of the sentences in the argument may turn out to be true or false. This enables us to explain very exactly what we mean by saying that, without fail, if the premises are true, then the conclusion is going to turn out to be true also. We interpret this to mean that in each possible case (in each of the cases 1 through 4), if the premises are true in that case, then the conclusion is true in that case. In other words, in all cases in which the premises are true, the conclusion is also true. In yet other words:

> To say that an argument (expressed with sentences of sentence logic) is *Valid* is to say that any assignment of truth values to sentence letters which makes all of the premises true also makes the conclusion true.

4-2. INVALIDITY AND COUNTEREXAMPLES

Let's look at an example of an *Invalid* argument (an argument which is not valid):

AvB
A
―――
B

	A	B	AvB	
*case 1	t	t	t	
*case 2	t	f	t	Counterexample
case 3	f	t	t	
case 4	f	f	f	

I have set up a truth table which shows the argument to be invalid. First I use a '*' to mark each case in which the premises are all true. In one of these cases (the second) the conclusion is false. This is what can't happen in a valid argument. So the argument is invalid. I will use the term *Counterexample* for a case which in this way shows an argument to be invalid. A counterexample to an argument is a case in which the premises are true and the conclusion is false.

In fact, we can use this idea of a counterexample to reword the definition of validity. To say that an argument is valid is to say that any assignment of truth values to sentence letters which makes all of the premises true also makes the conclusion true. We reword this by saying: An argument is valid just in case there is no possible case, no assignment of truth values to sentence letters, in which all of the premises are true and the conclusion is false. To be valid is to rule out any such possibility. We can break up this way of explaining validity into two parts:

> A *Counterexample* to a sentence logic argument is an assignment of truth values to sentence letters which makes all of the premises true and the conclusion false.
>
> An argument is *Valid* just in case there are no counterexamples to it.

Now let us reexpress all of this using sentences of sentence logic and the idea of logical truth. Let us think of an argument in which **X** is the conjunction of all the premises and **Y** is the conclusion. **X** and **Y** might be very complicated sentences. The argument looks like this:

$$\frac{X}{Y}$$

I will express an argument such as this with the words "**X.** Therefore **Y**".

A counterexample to such an argument is a case in which **X** is true and **Y** is false, that is, a case in which **X&~Y** is true. So to say that there are no possible cases in which there is a counterexample is to say that in all possible cases **X&~Y** is false, or, in all possible cases **~(X&~Y)** is true. But to say this is just to say that **~(X&~Y)** is a logical truth. The grand conclusion is that

> The argument "**X.** Therefore **Y**" is valid just in case the sentence **~(X&~Y)** is a logical truth.

4–3. SOUNDNESS

Logic is largely about validity. So to understand clearly what much of the rest of this book is about, you must clearly distinguish validity from some other things.

If I give you an argument by asserting to you something of the form "**X.** Therefore **Y**", I am doing two different things. First, I am asserting the premise or premises, **X.** Second, I am asserting to you that from these premises the conclusion, **Y** follows.

To see clearly that two different things are going on here, consider that there are two ways in which I could be mistaken. It could turn out that I

am wrong about the claimed truth of the premises, **X.** Or I could be wrong about the 'therefore'. That is, I could be wrong that the conclusion, **Y,** validly follows from the premises, **X.** To claim that **X** is true is one thing. It is quite another thing to make a claim corresponding to the 'therefore', that the argument is valid, that is, that there is no **possible** case in which the premises are true and the conclusion is false.

Some further, traditional terminology helps to emphasize this distinction. If I assert that the argument, "**X.** Therefore **Y**", is valid, I assert something about the relation between the premises and the conclusion, that in all lines of the truth table in which the premises all turn out true, the conclusion turns out true also. In asserting validity, I do not assert that the premises **are in fact** true. But of course, I can make this further assertion. To do so is to assert that the argument is not only valid, but *Sound:*

> An argument is *Sound* just in case, in addition to being valid, all its premises are true.

Logic has no special word for the case of a valid argument with false premises.

To emphasize the fact that an argument can be valid but not sound, here is an example:

Teller is ten feet tall or Teller has never taught logic.	AvB
Teller is not ten feet tall.	~A
Teller has never taught logic.	B

Viewed as atomic sentences, 'Teller is ten feet tall.' and 'Teller has never taught logic.' can be assigned truth values in any combination, so that the truth table for the sentences of this argument looks exactly like the truth table of section 4–1. The argument is perfectly valid. Any assignment of truth values to the atomic sentences in which the premises both come out true (only case 3) is an assignment in which the conclusion comes out true also. But there is something else wrong with the argument of the present example. In the real world, case 3 does not in fact apply. The argument's first premise is, in fact, false. The argument is valid, but not sound.

EXERCISES

4–1. Give examples, using sentences in English, of arguments of each of the following kind. Use examples in which it is easy to tell whether the premises and the conclusion are in fact (in real life) true or false.

a) A sound argument
b) A valid but not sound argument with a true conclusion
c) A valid but not sound argument with a false conclusion
d) An argument which is not valid (an *invalid* argument) all the premises of which are true
e) An invalid argument with one or more false premises

4–2. Use truth tables to determine which of the following arguments are valid. Use the following procedure, showing all your work: First write out a truth table for all the sentences in the argument. Then use a '*' to mark all the lines of the truth table in which all of the argument's premises are true. Next look to see whether the conclusion is true in the *ed lines. If you find any *ed lines in which the conclusion is false, mark these lines with the word 'counterexample'. You know that the argument is valid if and only if there are no counterexamples, that is, if and only if all the cases in which all the premises are true are cases in which the conclusion is also true. Write under the truth table whether the argument is valid or invalid (i.e., not valid).

a)	b)	c)	d)	e)
~(A&B)	~AvB	AvB	AvB	A
~A	A	~BvA	~AvB	Bv~C
~B	B	A	A	(A&B)v(A&~C)

4–3. Show that **X** is logically equivalent to **Y** if and only if the arguments "**X**. therefore **Y**" and "**Y**. Therefore **X**" are both valid.

4–4. THE CONDITIONAL

In section 4–2 we saw that the argument, "**X**. Therefore **Y**", is intimately related to the truth function ~(**X**&~**Y**). This truth function is so important that we are going to introduce a new connective to represent it. We will define **X⊃Y** to be the truth function which is logically equivalent to ~(**X**&~**Y**). You should learn its truth table definition:

	X	Y	X⊃Y
	t	t	t
Truth table	t	f	f
definition	f	t	t
of ⊃	f	f	t

Again, the connection between **X⊃Y** and the argument "**X**. Therefore **Y**" is that **X⊃Y** is a logical truth just in case the argument "**X**. Therefore **Y**" is valid.

Logicians traditionally read a sentence such as 'A⊃B' with the words 'If A, then B', and the practice is to transcribe 'If . . . then . . .' sentences of English by using '⊃'. So (to use a new example) we would transcribe 'If the cat is on the mat, then the cat is asleep.' as 'A⊃B'.

In many ways, this transcription proves to be problematic. To see why, let us forget '⊃' for a moment and set out afresh to define a truth functional connective which will serve as a transcription of the English 'If . . . then . . .':

	A	B	If A then B
case 1	t	t	
case 2	t	f	
case 3	f	t	
case 4	f	f	

(In the next two paragraphs, think of the example, 'If the cat is on the mat, then the cat is asleep.')

That is, by choosing t or f for each of the boxes under 'If A then B' in the truth table, we want to write down a truth function which says as closely as possible what 'If A then B' says in English.

The only really clear-cut case is case 2, the case in which the cat is on the mat but is not asleep. In this circumstance, the sentence 'If the cat is on the mat, then the cat is asleep.' is most assuredly false. So we have to put f for case 2 in the column under 'If A then B'. If the cat is both on the mat and is asleep, that is, if we have case 1, we may plausibly take the conditional sentence to be true. So let us put t for case 1 under 'If A then B'. But what about cases 3 and 4, the two cases in which A is false? If the cat is not on the mat, what determines whether or not the conditional, 'If the cat is on the mat, then the cat is asleep.', is true or false?

Anything we put for cases 3 and 4 is going to give us problems. Suppose we put t for case 3. This is to commit ourselves to the following: When the cat is not on the mat and the cat is asleep somewhere else, then the conditional, 'If the cat is on the mat, then the cat is asleep.', is true. But suppose we have sprinkled the mat with catnip, which always makes the cat very lively. Then, if we are going to assign the conditional a truth value at all, it rather seems that it should count as false. On the other hand, if we put f for case 3, we will get into trouble if the mat has a cosy place by the fire which always puts the cat to sleep. For then, if we assign a truth value at all, we will want to say that the conditional is true. Similar examples show that neither t nor f will always work for case 4.

Our problem has a very simple source: 'If . . . then . . .' in English can be used to say various things, many of which are not truth functional.

Whether or not an 'If . . . then . . .' sentence of English is true or false in these nontruth functional uses depends on more than just the truth values of the sentences which you put in the blanks. The truth of 'If you are five feet five inches tall, then you will not be a good basketball player.' depends on more than the truth or falsity of 'You are five feet five inches tall.' and 'You will not be a good basketball player.' It depends on the fact that there is some factual, nonlogical **connection** between the truth and falsity of these two component sentences.

In many cases, the truth or falsity of an English 'If . . . then . . .' sentence depends on a nonlogical connection between the truth and falsity of the sentences which one puts in the blanks. The connection is often causal, temporal, or both. Consider the claim that 'If you stub your toe, then it will hurt.' Not only does assertion of this sentence claim that there is some causal connection between stubbing your toe and its hurting, this assertion also claims that the pain will come **after** the stubbing. However, sentence logic is insensitive to such connections. Sentence logic is a theory only of truth functions, of connectives which are defined entirely in terms of the truth and falsity of the component sentences. So no connective defined in sentence logic can give us a good transcription of the English 'If . . . then . . .' in all its uses.

What should we do? Thus far, one choice for cases 3 and 4 seems as good (or as bad) as another. But the connection between the words '. . . therefore . . .' and 'If . . . then . . .' suggests how we should make up our minds. When we use 'If . . . then . . .' to express some causal, temporal, or other nonlogical connection between things in the world, the project of accurately transcribing into sentence logic is hopeless. But when we use 'If . . . then . . .' to express what we mean by '. . . therefore . . .' our course should be clear. To assert "**X.** Therefore **Y**", is to advance the argument with **X** as premise(s) and **Y** as conclusion. And to advance the argument, "**X.** Therefore **Y**", is (in addition to asserting **X**) to assert that the present case is not a counterexample; that is, it is to assert that the sentence $\sim$(**X**&$\sim$**Y**) is true. In particular, if the argument, "**X.** Therefore **Y**", is valid, there are no counterexamples, which, as we saw, comes to the same thing as $\sim$(**X**&$\sim$**Y**) being a logical truth.

Putting these facts together, we see that when "If **X** then **Y**" conveys what the 'therefore' in "**X.** Therefore **Y**" conveys, we can transcribe the "If **X** then **Y**" as $\sim$(**X**&$\sim$**Y**), for which we have introduced the new symbol **X**$\supset$**Y.** In short, when 'If . . . then . . .' can be accurately transcribed into sentence logic at all, we need to choose t for both cases 3 and 4 to give us the truth table for **X**$\supset$**Y** defined as $\sim$(**X**&$\sim$**Y**).

Logicians recognize that '$\supset$' is not a very faithful transcription of 'If . . . then . . .' when 'If . . . then . . .' expresses any sort of nonlogical connection. But since '$\supset$' agrees with 'If . . . then . . .' in the clear case 2 and the fairly clear case 1, '$\supset$' is going to be at least as good a transcription as any alternative. And the connection with arguments at least makes '$\supset$' the right choice for cases 3 and 4 when there is a right choice, that is, when 'If . . . then . . .' means '. . . therefore . . .'.

We have labored over the introduction of the sentence logic connective '$\supset$'. Some logic texts just give you its truth table definition and are done with it. But logicians use the '$\supset$' so widely to transcribe the English 'If . . . then . . .' that you should appreciate as clearly as possible the (truth functional) ways in which '$\supset$' does and the (nontruth functional) ways in which '$\supset$' does not correspond to 'If . . . then . . .'.

In these respects, the English 'and' and 'or' seem very different. 'And' and 'or' seem only to have truth functional aspects, so that they seem to correspond very closely to the truth functionally defined '&' and 'v'. Now that you have been through some consciousness raising about how English can differ from logic in having nontruth functional aspects, it is time to set the record straight about the 'and' and 'or' of English.

Surely, when I assert, 'Adam exchanged vows with Eve, and they became man and wife.' I do more than assert the truth of the two sentences 'Adam exchanged vows with Eve.' and 'They became man and wife.' I assert that there is a connection, that they enter into the state of matrimony **as a result of** exchanging vows. Similarly, if I yell at you, 'Agree with me or I'll knock your block off!' I do more than assert that either 'You will agree with me' or 'I will knock your block off' is true. I assert that nonagreement will produce a blow to your head. In these examples 'and' and 'or' convey some causal, intentional, or conventional association which goes above and beyond the truth functional combination of the truth values of the component sentences. 'And' can likewise clearly express a temporal relation which goes beyond the truth values of the components. When I say, 'Adam put on his seat belt and started the car.' I assert not only that 'Adam put on his seat belt.' and 'He started the car.' are both true. I also assert that the first happened before the second.

Although 'and', 'or', and 'If . . . then . . .' all have their nontruth functional aspects, in this respect 'If . . . then . . .' is the most striking. '$\supset$' is much weaker than 'If . . . then . . .', inasmuch as '$\supset$' leaves out all of the nontruth functional causal, temporal, and other connections often conveyed when we use 'If . . . then . . .'. Students sometimes wonder: If '$\supset$' (and '&' and 'v') are so much weaker than their English counterparts, why should we bother with them? The answer is that although truth functional sentence logic will only serve to say a small fraction of what we can say in English, what we can say with sentence logic we can say with profound clarity. In particular, this clarity serves as the basis for the beautifully clear exposition of the nature of deductive argument.

When the language of logic was discovered, its clarity so dazzled philosophers and logicians that many hoped it would ultimately replace English, at least as an all-encompassing exact language of science. Historically, it took decades to realize that the clarity comes at the price of important expressive power.

But back to '⊃'.

Here are some things you are going to need to know about the connective '⊃':

> A sentence of the form **X⊃Y** is called a *Conditional*. **X** is called its *Antecedent* and **Y** is called its *Consequent.*

Look at the truth table definition of **X⊃Y** and you will see that, unlike conjunctions and disjunctions, conditions are **not** symmetric. That is, **X⊃Y** is not logically equivalent to **Y⊃X.** So we need names to distinguish between the components. This is why we call the first component the antecedent and the second component the consequent (**not** the conclusion—a conclusion is a sentence in an argument).

Probably you will most easily remember the truth table definition of the conditional if you focus on the one case in which it is false, the one case in which the conditional always agrees with English. Just remember that a conditional is false if the antecedent is true and the consequent is false, and true in all other cases. Another useful way for thinking about the definition is to remember that if the antecedent of a conditional is false, then the whole conditional is true whatever the truth value of the consequent. And if the consequent is true, then again the conditional is true, whatever the truth value of the antecedent.

Finally, you should keep in mind some logical equivalences:

> The *Law of the Conditional* (C): **X⊃Y** is logically equivalent to ~(**X**&~**Y**) and (by De Morgan's law) to ~**X**v**Y.**
>
> The *Law of Contraposition* (CP): **X⊃Y** is logically equivalent to ~**Y**⊃~**X.**

4–5. THE BICONDITIONAL

We introduce one more connective into sentence logic. Often we will want to study cases which involve a conjunction of the form (**X⊃Y**)&(**Y⊃X**). This truth function of **X** and **Y** occurs so often in logic that we give it its own name, the *Biconditional,* which we write as **X≡Y.** Working out the truth table of (**X⊃Y**)&(**Y⊃X**) we get as our definition of the biconditional:

	X	Y	X≡Y
	t	t	t
Truth table	t	f	f
Definition of	f	t	f
	f	f	t

Because a biconditional has a symmetric definition, we don't have different names for its components. We just call them 'components'. You will remember this definition most easily by remembering that a biconditional is true if both components have the same truth value (both true or both false), and it is false if the two components have different truth values (one true, the other false). We read the biconditional **X≡Y** with the words '**X** if and only if **Y**'. With the biconditional, we get into much less trouble with transcriptions between English and sentence logic than we did with the conditional.

Given the way we define '≡', we have the logical equivalence:

> The *Law of the Biconditional* (B): **X≡Y** is logically equivalent to (**X⊃Y**)&(**Y⊃X**).

Remember that the conditional, **X⊃Y,** is a logical truth just in case the corresponding argument, "**X.** Therefore **Y**", is valid. Likewise, there is something interesting we can say about the biconditional, **X≡Y,** being a logical truth:

> **X≡Y** is a logical truth if and only if **X** and **Y** are logically equivalent.

Can you see why this is true? Suppose **X≡Y** is a logical truth. This means that in every possible case (for every assignment of truth values to sentence letters) **X≡Y** is true. But **X≡Y** is true only when its two components have the same truth value. So in every possible case, **X** and **Y** have the same truth value, which is just what we mean by saying that they are logically equivalent. On the other hand, suppose that **X** and **Y** are logically equivalent. This just means that in every possible case they have the same truth value. But when **X** and **Y** have the same truth value, **X≡Y** is true. So in every possible case **X≡Y** is true, which is just what is meant by saying that **X≡Y** is a logical truth.

EXERCISES

4–4. In section 1–6 I gave rules of formation and valuation for sentence logic. Now that we have extended sentence logic to include the connectives '⊃' and '≡', these rules also need to be extended. Write the full rules of formation and valuation for sentence logic, where sentence logic may now use all of the connectives '~', '&', 'v', '⊃', and '≡'. In your rules, also provide for three and more place conjunctions and disjunctions as described in section 3–2 in the discussion of the associative law.

4–5. Follow the same instructions as for exercise 4–2.

a) $\begin{array}{l} A\supset B \\ B \\ \hline A \end{array}$ b) $\begin{array}{l} A\supset\sim B \\ B \\ \hline \sim A \end{array}$ c) $\begin{array}{l} A\equiv B \\ A\vee\sim B \\ \hline B \end{array}$ d) $\begin{array}{l} A\equiv\sim B \\ A\vee\sim B \\ \hline A\vee B \end{array}$

e) $\begin{array}{c} (A\vee B)\supset(A\&C) \\ C\vee A \\ \hline \sim C \end{array}$ f) $\begin{array}{c} (A\vee B)\equiv(A\vee\sim C) \\ \sim B\vee C \\ \hline A\vee C \end{array}$

4–6. For each of the following sentences, establish whether it is a logical truth, a contradiction, or neither. Use the laws of logical equivalence in chapter 3 and sections 4–3 and 4–4, and use the fact that a biconditional is a logical truth if and only if its components are logically equivalent.

a) $(A\supset B)\equiv(\sim B\supset\sim A)$
b) $(A\equiv\sim A)\supset(B\equiv B)$
c) $A\equiv\sim A$
d) $A\supset\sim B$
e) $(A\supset B)\vee(A\supset\sim B)$
f) $\sim(A\vee B)\&(\sim A\supset B)$
g) $(A\equiv A)\supset(B\equiv\sim B)$
h) $[\sim(B\supset A)\&(C\supset A)]\supset(C\supset B)$
i) $[A\supset(B\supset C)]\supset[(A\supset B)\supset(A\supset C)]$

4–7. Discuss how you would transcribe 'unless' into sentence logic. Experiment with some examples, trying out the use of 'v', '⊃', and '≡'. Bear in mind that one connective might work well for one example, another connective for another example. As you work, pay attention to whether or not the compound English sentences you choose as examples are truth functional. Report the results of your research by giving the following:

a) Give an example of a compound English sentence using 'unless' which seems to be nontruth functional, explaining why it is not truth functional.
b) Give an example of a compound English sentence using 'unless' which seems to be truth functional, explaining why it is truth functional.
c) Give one example each of English sentences using 'unless' which can be fairly well transcribed into sentence logic using 'v', '⊃', '≡', giving the transcriptions into sentence logic.

4–8. Transcribe the following sentences into sentence logic, using the given transcription guide:

A: Adam loves Eve.
B: Adam is blond.
C: Eve is clever.
D: Eve has dark eyes.
E: Eve loves Adam.

a) If Eve has dark eyes, then Adam does not love her.
b) Adam loves Eve if she has dark eyes.
c) If Adam loves Eve, Eve does not love Adam.
d) Eve loves Adam only if he is not blond.
e) Adam loves Eve if and only if she has dark eyes.
f) Eve loves Adam provided he is blond.
g) Provided she is clever, Adam loves Eve.
h) Adam does not love Eve unless he is blond.
i) Unless Eve is clever, she does not love Adam.
j) If Adam is blond, then he loves Eve only if she has dark eyes.
k) If Adam is not blond, then he loves Eve whether or not she has dark eyes.
l) Adam is blond and in love with Eve if and only if she is clever.
m) Only if Adam is blond is Eve both clever and in love with Adam.

4–9. Consider the following four different kinds of nontruth functional connectives that can occur in English:

a) Connectives indicating connections (causal, intentional, or conventional)
b) Modalities (what must, can, or is likely to happen)
c) So-called "propositional attitudes," having to do with what people know, believe, think, hope, want, and the like
d) Temporal connectives, having to do with what happens earlier, later, or at the same time as something else.

Give as many English connectives as you can in each category. Keep in mind that some connectives will go in more than one category. ('Since' is such a connective. What two categories does it go into?) To get you started, here are some of these connectives: 'because', 'after', 'more likely than', 'Adam knows that', 'Eve hopes that'.

CHAPTER SUMMARY EXERCISES

Give brief explanations for each of the following. As usual, check your explanations against the text to make sure you get them right, and keep them in your notebook for reference and review.

a) Valid
b) Invalid
c) Counterexample
d) Sound
e) Conditional
f) Biconditional
g) Law of the Conditional
h) Law of Contraposition
i) Law of the Biconditional